Journal of Applied Mathematics and Mechanics 73 (2009) 134–144

Contents lists available at ScienceDirect

Journal of Applied Mathematics and Mechanics

journal homepage: www.elsevier.com/locate/jappmathmech

A dynamically consistent model of the contact stresses in the plane motion of a rigid body☆

A.P. Ivanov

Moscow, Russia

ARTICLE INFO

Article history:
Received 2 November 2008

ABSTRACT

The problem of determining dry friction forces in the case of the motion of a rigid body with a plane base over a rough surface is discussed. In view of the dependence of the friction forces on the normal load, the solution of this problem involves constructing a model of the contact stresses. The contact conditions impose three independent constraints on the kinematic characteristics, and the model must therefore include three free parameters, which are determined from these conditions at each instant. When the body is supported at three points, these parameters (for which the normal stresses can be taken) completely determine the model, while indeterminacy arises in the case of a larger number of contact points and, in order to remove this, certain physical hypotheses have to be accepted. It is shown that contact models consistent with the dynamics possess certain new qualitative properties compared with the traditional quasi-static models in which the type of motion of the body is not taken into account. In particular, a dependence of the principal vector and principal moment of the friction forces on the direction of sliding or pivoting of the body, as well as on the magnitude of the angular velocity, is possible.

It is will known[1–5] that, when the relative velocities at the points of the contact area are different, dry friction forces display certain properties, which are not exhibited by friction accompanying translational motion. Certain effects in the dynamics can only be explained[6–9] by a difference in the normal load distribution in the contact area from a static distribution.

1. Formulation of the problem and description of the model

Consider a rigid body with a plane base on a rough surface and let us introduce the inertial system of coordinates *OXYZ* with its origin in the bearing plane and with the *OZ* axis normal to this plane. The theorems on the change in momentum and angular momentum are expressed by the equalities

$$m\dot{\boldsymbol{\upsilon}} = \mathbf{F} + \mathbf{N} + \mathbf{T}, \quad (\mathbf{J}\boldsymbol{\Omega})^{\bullet} = \mathbf{M} + \mathbf{M}_N + \mathbf{M}_T \tag{1.1}$$

where υ is the velocity of the centre of mass of the body G, $\boldsymbol{\Omega}$ is the angular velocity, m and $\mathbf{J}$ are the mass and the central inertial tensor, $\mathbf{F}$, $\mathbf{N}$ and $\mathbf{T}$ are the principal vectors of the external forces, normal reaction and friction forces, and $\mathbf{M}$, $\mathbf{M}_N$ and $\mathbf{M}_T$ are the principal moments of the external forces, normal reaction and friction forces with respect to the centre of mass. The contact conditions require that the velocities (and accelerations) of all points of the body are parallel to the bearing plane. Consequently,

$$(\mathbf{F} + \mathbf{N}, \mathbf{k}) = 0, \quad \boldsymbol{\Omega} = \omega\mathbf{k} \tag{1.2}$$

where $\mathbf{k}$ is the unit vector along the z coordinate and $\omega = \omega(t)$ is a certain scalar function. When account is taken of equalities (1.2), the second formula of (1.1) takes the form

$$\dot{\omega}\mathbf{J}\mathbf{k} + \omega^2\mathbf{k} \times (\mathbf{J}\mathbf{k}) = \mathbf{M} + \mathbf{M}_N + \mathbf{M}_T \tag{1.3}$$

☆ *Prikl. Mat. Mekh.* Vol. 73, No. 2, pp. 189–203, 2009.

E-mail address: apivahov@orc.ru.

0021-8928/$ – see front matter
doi:10.1016/j.jappmathmech.2009.04.003

The reaction of the support in formulae (1.1) is calculated using the formulae (henceforth, unless otherwise indicated, integration is carried out over a domain D consisting of the points of the body which are in contact with the support)

$$\mathbf{N} = \mathbf{k}\iint n(A)ds, \quad \mathbf{M}_N = \iint n(A)\mathbf{r}(A)\times\mathbf{k}ds$$
$$\mathbf{T} = \iint \mathbf{t}(A)ds, \quad \mathbf{M}_T = \iint \mathbf{r}(A)\times\mathbf{t}(A)ds; \quad \mathbf{r}(A) = \mathbf{GA} \tag{1.4}$$

where $n(A)$ and $\mathbf{t}(A)$ are the normal and shear stress at the point $A\in D$. The friction is locally described by the Amonton–Coulomb law

$$\mathbf{t}(A) = -\mu n(A)\frac{\boldsymbol{\upsilon}(A)}{|\boldsymbol{\upsilon}(A)|} \tag{1.5}$$

where $\upsilon(A)$ is the body velocity at the point A and μ is the coefficient of friction.

Euler's formula

$$\boldsymbol{\upsilon}(A) = \boldsymbol{\upsilon} + \boldsymbol{\Omega}\times\mathbf{r}(A)$$

is used to determine the vector $\upsilon(A)$.

In this discussion, the key problem is to determine of the distribution $n(A)$. One contact model or another, constructed without taking account of the motion of the body and the action of friction forces, is conventionally used to solve it. A uniform distribution, a Hertz contact law and others are among a number of such quasi-static models. The drawback of such an approach lies in its incompatibility with the fundamental theorems of dynamics, which are expressed by formulae (1.1).

As an example, we will consider a heavy body on a horizontal plane. In the rest state, the normal reactions have a resultant which is opposite to the force of gravity. If, however, the body slides inertially over the support, then the friction forces lying in the bearing plane cannot be compensated by the forces of inertia, gravity and the normal reaction, which have a resultant applied at the centre of mass. This contradiction can only be assumed to be unimportant in the case when the height of the centre of mass above the support is negligibly small, that is, the body is a plate. The motion of the stone in the game of curling can serve as another well-known example: being curled, the stone moves along an arc-shaped trajectory which enables a skilful player to bend around a "defender". This effect can only be explained by the non-uniformity of the normal load distribution in the domain of contact,[6–9] which lies beyond the limits of a quasi-static model. Note that the distortion of the trajectories of asymmetric bodies can also be explained without taking account of dynamic deformations.[10]

Equalities (1.2) impose three constraints on the distribution of the normal load $n(A)$. A dynamically consistent model must therefore include at least three free parameters which can be chosen in accordance with these constraints:

$$n(A) = n(x, y, \lambda), \quad \lambda\in\mathbb{R}^3 \tag{1.6}$$

where x and y are the coordinates of the point A. Substituting the function (1.6) into system (1.2) - (1.4), we obtain three equations for determining λ. This enables us to determine the reactions in the equations of motion (1.1).

An example of this approach is the linear model

$$n(A) = \lambda_0 + \lambda_1 x + \lambda_2 y \tag{1.7}$$

the basis of which is the assumption that the body is absolutely rigid and the plane undergoes small deformations, generating normal stresses, as given by Hooke's law. Formula (1.7) also describes the load distribution in the case of a discrete three-point contact (in this case, integrals (1.3) are replaced by sums of three terms); additional physical hypotheses are not required here. This model has previously been used in conjunction with a quasi-static approach,[11] that is, the coefficients λ_j ($j = 0, 1, 2$) were calculated in the rest state.

The aim of this paper is to study frictional properties within the limits of model (1.7) in which the coefficients λ_j are determined at each instant from conditions (1.2).

2. Determination of the parameters of the linear model

From the physical point of view, the contact between a body and a support is unilateral, that is, the inequality

$$n(A)\geq 0, \quad A\in D \tag{2.1}$$

is satisfied at all points of the contact area.

The vector equality (1.3), in conjunction with the first condition of (1.2), taking account of formulae (1.4) and (1.7), constitutes a system of linear algebraic equations in the unknowns $\dot{\omega}$ and λ_j ($j = 0, 1, 2$). The calculation of the parameters λ_j can be simplified by using an auxiliary system of coordinates with origin O' at the centre of mass of the domain D, which is regarded as a homogeneous plate with axes parallel to the axes of the basic system $OXYZ$ (we retain the previous notation for the coordinates). In this case, the first of formulae of (1.4) takes the form

$$\mathbf{N} = \lambda_0 S(D)\mathbf{k} \tag{2.2}$$

where $S(D)$ is the area of the domain. Substituting this expression into the first formula of (1.2), we obtain

$$\lambda_0 = (-(\mathbf{F}, \mathbf{k}))/S(D) \tag{2.3}$$

that is, the quantity λ_0 is equal to the normal pressure in the case of a uniform load distribution.

In the general case, the principal central axes of inertia of the body are not parallel to the coordinate axes and the vector **Jk** is not collinear with **k**. In order to eliminate the unknown $\dot{\omega}$ from Eq. (1.3), we project it onto the unit vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ which are orthogonal to **Jk**. Finally, we obtain the following second order linear algebraic system in the unknowns λ_1 and λ_2:

$$\omega^2(\mathbf{e}_n \times \mathbf{k}, \mathbf{Jk}) = (\mathbf{M}, \mathbf{e}_n) + \iint \Psi_n(A)(\lambda_0 + \lambda_1 x + \lambda_2 y)ds, \quad n = 1, 2$$
$$\Psi_n(A) = \left(\mathbf{k} - \mu\frac{\boldsymbol{\upsilon}(A)}{|\boldsymbol{\upsilon}(A)|}, \mathbf{e}_n \times \mathbf{r}(A)\right), \quad \mathbf{r}(A) = (x - x_G, y - y_G, -h) \tag{2.4}$$

where h is the distance from the centre of mass to the support. System (2.4) has a unique solution in the non-degenerate case. This solution will be permissible if it satisfies condition (2.1). Violation of this condition means that either the body detaches from the support (it flips) or the actual contact area is only a part of the domain D. In the latter case, the problem of determining the reactions is extremely complicated in view of the non-linearity, and its analytical solution is only possible in certain special cases.[11]

After determining the parameters λ_j, the angular acceleration $\dot{\omega}$ can be found by multiplying equality (1.3) by the vector **k**:

$$\dot{\omega}(\mathbf{Jk}, \mathbf{k}) = (\mathbf{M}, \mathbf{k}) + (\mathbf{M}_T, \mathbf{k}) \tag{2.5}$$

3. Determination of the friction for a heavy solid of revolution on a horizontal support

In the case of a solid of revolution, the domain D takes the form of a circle of radius a and the axis of symmetry is parallel to the OZ axis. We will assume that the plane is horizontal and that there are no external forces other than gravitational forces. From symmetry considerations, the vector **Jk** is collinear with **k** and, therefore, the left-hand sides in Eq. (2.4) are equal to zero. Furthermore, $\mathbf{M} = 0$, $x_G = y_G = 0$ and the basis unit vectors **i** and **j** can be taken as $\mathbf{e}_n$, directing the ordinate axis in the direction of the velocity of the centre of the circle. We obtain the expressions

$$\Psi_1 = y - \mu h(\upsilon + \omega x)\tilde{q}, \quad \Psi_2 = -x - \mu h\omega y\tilde{q}$$
$$\tilde{q} = (\omega^2 y^2 + (\upsilon + \omega x)^2)^{-1/2} \tag{3.1}$$

for the quantities Ψ_n in formulae (2.4).

Substituting these into Eqs. (2.4), we obtain

$$a_{11}\lambda_1 + a_{12}\lambda_2 = a_{10}\lambda_0, \quad a_{21}\lambda_1 + a_{22}\lambda_2 = a_{20}\lambda_0$$
$$a_{11} = -\mu h\iint x(\upsilon + \omega x)\tilde{q}ds, \quad a_{12} = \iint y^2 ds - \mu h\iint y(\upsilon + \omega x)\tilde{q}ds = \iint y^2 ds = \frac{1}{4}\pi a^4$$
$$a_{21} = -\iint x^2 ds - \mu h\iint \omega xy\tilde{q}ds = -\iint x^2 ds = -\frac{1}{4}\pi a^4, \quad a_{22} = -\mu h\omega\iint y^2\tilde{q}ds,$$
$$a_{10} = \mu h\iint(\upsilon + \omega x)\tilde{q}ds, \quad a_{20} = \iint(x + \mu h\omega y\tilde{q})ds = 0 \tag{3.2}$$

Here, the symmetry of the domain D with respect to the coordinate axes and the oddness of the integrands has been used when simplifying the integrals.

It follows from these formulae that the coefficients a_{12}, a_{21} and a_{20} have constant values regardless of the form of motion and the magnitude of μ. These values will not be indicated in the subsequent treatment in this section.

The coefficient a_{12} is positive and the remaining coefficients of the unknowns are negative in the case when $\omega > 0$. Consequently,

$$\Delta = a_{11}a_{22} - a_{12}a_{21} > 0 \tag{3.3}$$

and system (3.2) has a unique solution

$$\lambda_1 = a_{10}a_{22}\lambda_0/\Delta, \quad \lambda_2 = -a_{10}a_{21}\lambda_0/\Delta \tag{3.4}$$

We will now consider some special cases of formulae (3.4):

1) if $\upsilon = 0$ (pure curling), then $a_{10} = 0$, whence it follows that $\lambda_1 = \lambda_2 = 0$, that is, the normal load distribution is uniform in the contact zone;
2) if $\omega = 0$ (pure sliding), then $a_{22} = 0$, whence $\lambda_1 = 0$, $\lambda_2 = a_{10}\lambda_0/a_{12}$. Further, $\tilde{q} = 1/\upsilon$ and, consequently, $a_{10} = \pi\mu h a^2$ and the normal load distribution is described by the formula

$$n(A) = \lambda_0(1 + 4\mu h a^{-2} y) \tag{3.5}$$

According to formula (3.5), the normal pressure increases from the centre of the circle D in the direction of slipping. The normal pressure non-negativity condition (2.1) is satisfied if the coefficient of friction is sufficiently small, so that the inequality

$$4\mu h \le a \tag{3.6}$$

is satisfied.

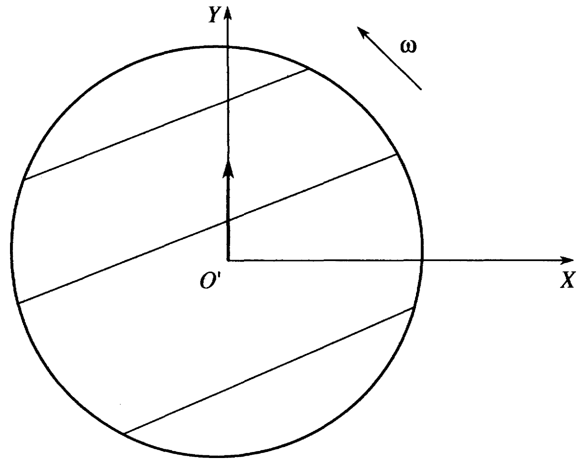

Fig. 1.

Note that, in order to maintain the contact zone as the body slips (that is, it does not turn over), the moment of the normal reaction, with respect to the centre of mass, concentrated at the front point of the base, must not be less than the moment of the friction forces, that is,

$$\mu h \leq a \tag{3.7}$$

For the model being discussed, intermediate values of the coefficient of friction, for which the inequality (3.7) is satisfied and (3.6) is not, correspond to contact in a part of the domain D.

Property 1. If $\upsilon > 0$, $\omega > 0$, then

$$\lambda_1 < 0, \quad \lambda_2 > 0$$

This property follows from the inequality $a_{10} > 0$ and equalities (3.4). It means that, as a body which has been curled anticlockwise slides, there is a tilting to the left together with forward pitching. The lines of equal pressure are a family of parallel straight lines (Fig. 1)

$$\lambda_1 x + \lambda_2 y = \text{const}$$

Note that the effect of the distortion of a trajectory has previously been explained either by forward pitching[6,9] or by tilting.[8]

We will now determine the force and the moment of the friction forces using formulae (1.3) with the function (1.7):

$$\mathbf{T} = -\mu\iint(\lambda_0 + \lambda_1 x + \lambda_2 y)(-\omega y,\ \upsilon + \omega x)\tilde{q}ds$$

$$\mathbf{M}_T = -\mu\omega\iint(\lambda_0 + \lambda_1 x + \lambda_2 y)(x(x+\delta) + y^2)\tilde{q}dxdy, \quad \delta = \frac{\upsilon}{\omega} \tag{3.8}$$

Since the domain of integration is symmetrical about the axes, in formula (3.8) the integrals of terms which are odd in y vanish. Consequently,

$$T_X = \mu\lambda_2\omega\iint y^2\tilde{q}ds, \quad T_Y = -\mu\omega\iint(\lambda_0 + \lambda_1 x)(x+\delta)\tilde{q}ds$$

$$\mathbf{M}_T = -\mu\omega\iint(\lambda_0 + \lambda_1 x)(x(x+\delta) + y^2)\tilde{q}dxdy \tag{3.9}$$

The coefficients λ_1 and λ_2 in formulae (3.9) are defined by relations (3.4). In order to evoluate integrals (3.9), we can change to polar coordinates ρ and φ using the formulae

$$x + \delta = \rho\cos\varphi, \quad y = \rho\sin\varphi \tag{3.10}$$

and then consider three cases depending on the position of the instantaneous centre of rotation with respect to the contact area. Similar calculations have been carried out[11] in the problem of the motion of an asymmetric body with a circular base (in a quasi-static formulation) and expressions for the friction forces and moments have been obtained in terms of complete elliptic integrals. We shall therefore confine ourselves to a qualitative analysis of these formulae.

Property 2. The value of T_X is non-zero which indicates that the body deviates to the right (when it is curled anticlockwise). By virtue of equalities (3.4), for small values of μ the coefficient λ_2 is approximately proportional to μ and, therefore, $T_X = O(\mu^2)$, that is, $T_X|T| = O(\mu)$ and the effect is amplified as the coefficient of friction increases.

Property 3. The quantity T_Y consists of two components. The first of these is negative and is identical in form with the friction force acting on a body in the case of a uniform normal load distribution. The second component, which takes account of the non-uniformity of the above-mentioned distribution, has the opposite sign.

Property 4. The moment of the friction forces $\mathbf{M}_T$ also consists of two components, the first of which is identical in form with the moment of the friction forces acting on a body in the case of a uniform normal load distribution and the second has the opposite sign.

We will now consider special cases of formula (3.9).

If $\delta \to 0$, then, for non-constant coefficients of system (3.2), we find, up to terms of the order of δ^2

$$a_{11} = -\frac{1}{3}\mu h \int_0^{2\pi} \cos^2\varphi q^{*3} d\varphi = -\frac{\pi}{3}\mu h a^3, \quad a_{22} = -\frac{1}{3}\mu h \int_0^{2\pi} \sin^2\varphi q^{*3} d\varphi = -\frac{\pi}{3}\mu h a^3$$

$$a_{10} = \mu h \delta \int_0^{2\pi} \cos^2\varphi q^* d\varphi = \pi\mu h a\delta; \quad q^* = \sqrt{a^2 - \delta^2 \sin^2\varphi}$$

and we obtain its solution in the form

$$\lambda_1 = -\frac{4}{3}\kappa\lambda_2 = -\frac{48\kappa^2}{9+16\kappa^2}\frac{\lambda_0\delta}{a^2}, \quad \kappa = \frac{\mu h}{a} \tag{3.11}$$

Formulae (3.9) then take the form

$$T_X = \mu N \frac{\delta}{a}\frac{12\kappa^2}{9+16\kappa^2}, \quad T_Y = -\mu N \frac{\delta}{a}\left(1 - \frac{16\kappa^2}{9+16\kappa^2}\right) \tag{3.12}$$

When $\delta \to 0$, the expression for $\mathbf{M}_T$ is identical with the magnitude of the moment calculated for a uniform distribution. According to condition (3.6), $\kappa \le 1/4$ in the case of complete contact. Consequently, the magnitude of T_X constitutes not more than 7.5% of the maximum friction force (calculated for a uniform distribution) and the magnitude of T_Y constitutes no less than 90% of this maximum.

In the other limiting case $\delta \to \infty$, we carry out calculations up to terms of the third order in $\varepsilon = \delta^{-1}$. For system (3.2), we obtain

$$\tilde{q} = \frac{1}{\upsilon}\left(1 - \varepsilon x + \varepsilon^2 x^2 - \frac{1}{2}\varepsilon^2 y^2\right)$$

$$a_{11} = 0, \quad a_{22} = -\frac{1}{4}\pi\varepsilon\mu h a^4, \quad a_{10} = \pi\mu h a^2\left(1 - \frac{1}{8}\varepsilon^2 a^2\right)$$

Consequently,

$$\lambda_1 = -4\varepsilon\kappa^2\lambda_0, \quad \lambda_2 = \mu h(8a^{-2} - \varepsilon^2)\lambda_0/2 \tag{3.13}$$

Substituting expressions (3.13) into integrals (3.9), we obtain

$$T_X = \varepsilon\mu^2 h N, \quad T_Y = -\mu N\left(1 - \frac{1}{4}\varepsilon^2 a^2\right)$$

$$M_T = -\mu N \varepsilon a^2\left(\frac{1}{4} - \kappa^2\right) + O(\varepsilon^2) \tag{3.14}$$

When account is taken of constraint (3.6), formula (3.14) shows that the presence of a non-uniformity in the normal load distribution leads to a reduction in the friction moment by no more than 25%.

If $\delta = a$, the integrals in formulae (3.2) can be evaluated in terms of elementary functions. In the polar coordinates (3.10), the domain of integration is described by the formulae

$$0 \le \rho \le 2a\cos\varphi, \quad -\pi/2 \le \varphi \le \pi/2$$

and $\tilde{q} = (\omega\rho)^{-1}$. Hence, we obtain

$$a_{11} = -\mu h \int_{-\pi/2}^{\pi/2} \cos\varphi d\varphi \int_0^{2a\cos\varphi} \rho(\rho\cos\varphi - a)d\rho = -\frac{8}{45}\kappa a^4$$

$$a_{22} = -\frac{8}{3}\mu h a^3 \int_{-\pi/2}^{\pi/2} \sin^2\varphi\cos^3\varphi d\varphi = -\frac{32}{45}\kappa a^4$$

$$a_{10} = \mu h \int_{-\pi/2}^{\pi/2} 2a^2\cos^3\varphi d\varphi = \frac{8}{3}\kappa a^3 \tag{3.15}$$

(the remaining coefficients in system (3.2) are constants). The solution of system (3.2) with the coefficients (3.15) has the form

$$\lambda_1 = C_1\lambda_0, \quad \lambda_2 = C_2\lambda_0$$

$$C_2 = \frac{8}{3}\frac{\kappa}{a}\left(\frac{1024}{2025\pi}\kappa^2 + \frac{1}{4}\pi\right)^{-1}, \quad C_1 = -\frac{128}{45\pi}\kappa C_2$$

Calculations using formulae (3.9) lead to the following results

$$T_X = \frac{32}{45}\mu a^3\lambda_2, \quad T_Y = -\mu\left(\frac{8}{3}a^2\lambda_0 + \frac{8}{45}a^3\lambda_1\right)$$

$$M_T = -\mu\left(\frac{8}{9}a^3\lambda_0 + \frac{8}{15}a^4\lambda_1\right)$$

When account is taken of constraint (3.6), non-uniformity in the normal load distribution leads, in this case, to a reduction in the magnitude of T_Y by no more than 1% and in the magnitude of the moment M_T by no more than 12%.

4. Determination of the reactions of the supports for a body with three points of support

We will now investigate the case when the contact area consists of three non-collinear points C_1, C_2, C_3 (a tripod). We will assume that the plane is horizontal and that there are no external forces apart from gravity. This system is statically determinable, that is, for a stationary body, the normal reactions at the contact points N_s ($s = 1, 2, 3$) are uniquely determined from the contact conditions (1.2).

In order to solve the problem of determining the forces acting on a moving body, we will use of the method described in Sections 1 - 3. We take the normal reactions at the points of contact $N_s \geq 0$ as the parameters of the model λ_s ($s = 1, 2, 3$) and, by analogy with Eqs. (2.4), to determine these reactions we have the system

$$\sum N_s(\mathbf{e}_n \times \mathbf{r}_s, \mathbf{k} - \mu\mathbf{t}_s) = \omega^2(\mathbf{e}_n \times \mathbf{k}, \mathbf{Jk}), \quad n = 1, 2; \quad \mathbf{t}_s = \frac{\boldsymbol{\upsilon}(C_s)}{|\boldsymbol{\upsilon}(C_s)|} \tag{4.1}$$

$$\sum N_s = P \tag{4.2}$$

where P is the weight of the body. Summation is henceforth carried out from $s = 1$ to $s = 3$.

Definition. We shall say that the tripod is balanced if one of its three principal central axes of inertia is vertical.

The mathematical formulation of this property is that $\mathbf{k}$ is an eigenvector of the central inertia tensor. In particular, this property is always satisfied in the case of a plane body (a plate) (Ref. 12, Section 319).

In the case of a balanced body, the left-hand side of equalities (1.4) is equal to zero:

$$\sum N_s(\mathbf{e}_n \times \mathbf{r}_s, \mathbf{k} - \mu\mathbf{t}_s) = 0, \quad n = 1, 2 \tag{4.3}$$

and the vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ are horizontal. This system takes a particularly simple form in the case of a plate. The vectors $\mathbf{r}_s$ are then also horizontal and the terms $\mu\mathbf{t}_s$ in system (4.3) therefore disappear. Consequently, the normal load distribution is independent of the form of the motion of the plate. If the height of the centre of mass above the support $h \neq 0$, then the analogous assertion only holds when there is no friction.

Proposition 1. In the case when the tripod is balanced but there is no friction, the normal load distribution is independent of the form of the motion. The criterion for the reactions N_s ($s = 1, 2, 3$) to be positive is that the centre of mass is projected into a point G' which lies within the triangle $C_1C_2C_3$.

The first part of this proposition follows from the fact that the quantity ω^2 does not occur in system (4.3), (4.2). The last assertion expresses one of the well-known results of statics (Ref. 12, Section 112). Explicit formulae can be obtained for the reactions by equating the resultant moments of the force of gravity and one of the reactions with respect to an axis passing through the opposite contact points to zero. We denote the height of the triangle $C_1C_2C_3$, dropped from the corresponding vertex, by h_s ($s = 1, 2, 3$) and the distances from a point G' to the sides of the triangle by d_s. Then,

$$N_s = \frac{d_s}{h_s}P, \quad s = 1, 2, 3 \tag{4.4}$$

Another geometrical interpretation of equalities (4.4) is: by joining the point G' to the vertices of the triangle $C_1C_2C_3$, we obtain three triangles, the areas of which are proportional to the normal loads at the corresponding vertices.

Proposition 2. In the case of an unbalanced tripod (when there is no friction), the normal load distribution depends on the angular velocity. Motion without breaking contact with the plane is possible for values of the angular velocity modulus in a certain finite interval $|\omega| \in (\omega_1, \omega_2)$ which can be empty.

Proof. From Eq. (4.1) when $n = 1$ and $n = 2$, we subtract Eq. (4.2) with coefficients such that the right-hand sides of Eq. (4.1) vanish. We obtain

$$\sum N_s\left(\mathbf{k}\times\mathbf{e}_n, \mathbf{r}_s + \frac{\omega^2}{P}\mathbf{J}\mathbf{k}\right) = 0, \quad n = 1, 2 \tag{4.5}$$

System (4.5), (4.2) has the same form as in the case of a balanced tripod, only all the contact points are shifted by the value of the horizontal component of the vector $\omega^2 P^{-1}\mathbf{Jk}$. Proposition 1 can therefore be used. All such shifts have a single direction but different moduli for different values of ω which also proves this proposition.

We will now discuss the case when $\mu \neq 0$. The dynamic analogy between the additional forces and the shifts of the vertices of the support triangle can also be used here.

Proposition 3. In the case of a balanced tripod when there is friction, the normal load distribution can be determined using formulae (4.4), applied to the auxiliary triangle $\bar{C}_1\bar{C}_2\bar{C}_3$ which is obtained from the supporting triangle $C_1C_2C_3$ by shifting each of the vertices in a direction opposite to the direction in which it slides by an amount $\delta = \mu_h$ (h is the height of the centre of mass G above the support). The criterion for the reactions N_s ($s = 1, 2, 3$) to be positive is that the point G' lies within the triangle $\bar{C}_1\bar{C}_2\bar{C}_3$.

Proof. In the case under discussion, the vectors $\mathbf{e}_n$ in system (4.3), (4.2) are horizontal. We solve the equation

$$(\mathbf{e}_n\times\mathbf{r}_s, \mathbf{k} - \mu\mathbf{l}) = (\mathbf{e}_n\times(\mathbf{r}+\boldsymbol{\delta}), \mathbf{k}) \tag{4.6}$$

where $\mathbf{l}$ is a specified horizontal vector with respect to an unknown horizontal vector $\boldsymbol{\delta}$. After simplification, we obtain the relation

$$-\mu h(\mathbf{e}_n\times\mathbf{k}, \mathbf{l}) = (\mathbf{e}_n\times\mathbf{k}, \boldsymbol{\delta})$$

from which $\delta = \mu h\mathbf{l}$. Successively putting $\mathbf{l} = \mathbf{t}_s$ ($s = 1, 2, 3$), we substitute expressions (4.6) into Eqs. (4.3) and then make use of Proposition 1.

Proposition 4. In the case of an unbalanced tripod when there is friction, it is also possible to determine the normal load distribution using formulae (4.2), (4.3), applied to the auxiliary triangle $\bar{C}_1\bar{C}_2\bar{C}_3$, obtained from the supporting triangle $C_1C_2C_3$ by an appropriate shift.

Proof. Without loss of generality, we put

$$\mathbf{e}_1 = \mathbf{i}, \quad \mathbf{e}_2 = \alpha\mathbf{j} + \beta\mathbf{k}, \quad \alpha^2 + \beta^2 = 1 \tag{4.7}$$

where the numbers α and β are positive. Then, $\mathbf{e}_1 \times \mathbf{k} = -\mathbf{j}$, $\mathbf{e}_2 \times \mathbf{k} = \alpha\mathbf{i}$). Equation (4.6) is transformed to the form

$$\begin{aligned}(\boldsymbol{\delta}, \mathbf{j}) &= -\mu h(\mathbf{l}, \mathbf{j}) \text{ when } n = 1\\ \alpha(\boldsymbol{\delta}, \mathbf{i}) &= \mu(\mathbf{e}_2\times\mathbf{r}, \mathbf{l}) \text{ when } n = 2\end{aligned} \tag{4.8}$$

Since $\alpha > 0$, equalities (4.8) uniquely define the vector $\boldsymbol{\delta}$ in formula (4.5). Furthermore, an additional equal shift of all of the vertices of the triangle is necessary to compensate the left-hand sides in formulae (4.3) (see Proposition 2). The assertion is proved.

Corollary. The problem of determining the normal reactions from Eqs. (4.3) a, (4.2) has a unique admissible solution if and only if the point G' lies within the triangle $\bar{C}_1\bar{C}_2\bar{C}_3$.

Having determined the reactions, it is then possible to find the distribution of the accelerations from system (1.1):

$$m\dot{\boldsymbol{\upsilon}} = \mathbf{T}, \quad \dot{\omega}(\mathbf{Jk}, \mathbf{k}) = (\mathbf{M}_T, \mathbf{k}); \quad \mathbf{T} = -\mu\sum\frac{\boldsymbol{\upsilon}_s}{|\boldsymbol{\upsilon}_s|}N_s, \quad \mathbf{M}_T = -\mu\sum\frac{\mathbf{r}_s\times\boldsymbol{\upsilon}_s}{|\boldsymbol{\upsilon}_s|}N_s \tag{4.9}$$

5. Translational motions of a tripod and motions close to them

We will investigate which simple motions exist in the system being considered.

In the case of translational motion, we have $\omega = 0$ and, then,

$$\boldsymbol{\upsilon}_s = \boldsymbol{\upsilon}, \quad s = 1, 2, 3 \tag{5.1}$$

In Eq. (1.3) $\omega = 0$, $t_s = t = \upsilon/|\upsilon|$, and, hence, we obtain

$$\mathbf{M}_N + \mathbf{M}_T = \sum N_s\mathbf{r}_s\times(\mathbf{k}-\mu\mathbf{t}) = 0 \tag{5.2}$$

This means that the vector $\Sigma N_s\mathbf{r}_s$ is a linear combination of the vectors $\mathbf{k}$ and $\mathbf{t}$. Therefore,

$$(\mathbf{M}_T, \mathbf{k}) = \sum(\mathbf{r}_s\times\mathbf{T}_s, \mathbf{k}) = -\mu(\sum N_s\mathbf{r}_s\times\mathbf{t}, \mathbf{k}) = 0 \tag{5.3}$$

and, by virtue of the second equation of (4.9), we have $\dot{\omega} = 0$.

Hence, a tripod can execute translational motion in any direction only if the normal reactions at the supports are negative at the same time. As a consequence of Propositions 3 and 4, in order to check this condition it is necessary to construct the auxiliary triangle $\bar{C}_1\bar{C}_2\bar{C}_3$ and then to investigate whether the point G' lies within this triangle. This problem is particularly simple to solve in the case of a balanced tripod: the initial triangle $C_1C_2C_3$ is shifted translationally by the vector $\delta = -mh\mathbf{t}$.

We shall say that a motion is *close to translational* if

$$|\omega|\max_s\left|\mathbf{r}_s^0\right| \ll \upsilon$$

We define the system of coordinates in such a way that the velocity of the centre of mass is directed along the abscissa and, then,

$$\boldsymbol{\upsilon}_s = (\upsilon - \omega y_s)\mathbf{i} + \omega x_s \mathbf{j}, \quad s = 1, 2, 3$$

whence

$$\boldsymbol{\upsilon}_s/|\boldsymbol{\upsilon}_s| = \mathbf{i} + \varepsilon x_s \mathbf{j} + O(\varepsilon^2), \quad \varepsilon = \omega/\upsilon \tag{5.4}$$

Substituting these expressions into the second formula of (4.9), we obtain

$$\dot{\omega}(\mathbf{Jk}, \mathbf{k}) = \mu \sum N_s(\mathbf{r}_s^0, \mathbf{k} \times (\mathbf{i} + \varepsilon x_s \mathbf{j})) = \mu \sum N_s(y_s - \varepsilon x_s^2) \tag{5.5}$$

In order to determine the normal load distribution, we use system (4.5), (4.2), where the square of the angular velocity is not taken into account in the first approximation. If the tripod is balanced, then, by putting $\mathbf{e}_1 = \mathbf{i}$, $\mathbf{e}_2 = \mathbf{j}$, we obtain

$$\sum N_s(\mathbf{r}_s, (\mathbf{k} - \mu(\mathbf{i} + \varepsilon x_s \mathbf{j})) \times \mathbf{i}) = 0$$

$$\sum N_s(\mathbf{r}_s, (\mathbf{k} - \mu(\mathbf{i} + \varepsilon x_s \mathbf{j})) \times \mathbf{j}) = 0, \quad \sum N_s = P$$

whence, after simplifications, we have the relations

$$\sum N_i(y_i - \mu h x_i) = 0, \quad \sum N_i x_i = \mu h P \tag{5.6}$$

and, taking account of these, we write formula (5.5) in the form

$$\dot{\omega}(\mathbf{Jk}, \mathbf{k}) = -\mu \frac{\omega}{\upsilon} \sum N_s (x_s - \mu h)^2 \tag{5.7}$$

The following assertion is proved.

Proposition 5. In the case of the motions of a balanced tripod which are close to translational, the friction moment is of the first order of smallness with respect to the angular velocity and its absolute magnitude is independent of the direction of curling.

Corollary. The rate of curling decreases in absolute magnitude for any direction of sliding.

We will now consider the case of an unbalanced tripod. We obtain the relations

$$\sum N_s(y_s - \mu \varepsilon h x_s) = \dot{\omega}(\mathbf{Jk}, \mathbf{i}), \quad \sum N_s x_s = \mu h P - \dot{\omega}(\mathbf{Jk}, \mathbf{j}) \tag{5.8}$$

in the same way as Eqs (5.6).

As was shown above, $\dot{\omega} = 0$ when $\omega = 0$ and, therefore $\dot{\omega} = O(\varepsilon)$ in formulae (5.8) whence

$$\sum N_s(y_s - \mu \varepsilon h x_s) = \dot{\omega}(\mathbf{Jk}, \mathbf{i}) + \mu^2 \varepsilon h^2 P + O(\varepsilon^2) \tag{5.9}$$

Subtracting equality (5.9) from (5.5), we obtain

$$\dot{\omega}(\mathbf{Jk}, \mathbf{k} - \mu \mathbf{i}) = -\mu \frac{\omega}{\upsilon} \sum N_s (x_s - \mu h)^2 + O(\omega^2) \tag{5.10}$$

Unlike in the case of a balanced tripod, the coefficient of $\dot{\omega}$ on the left-hand side of formula (5.10) can be negative. For example, in the case of very prolate solids of revoluation with an axis close to the vertical, the vectors **Jk** and **k** can make an angle which is close to a right angle. If the direction of sliding coincides with the horizontal projection of the vector **Jk**, this situation arises even for small values of the coefficient of friction.

We will resume the above analysis.

Proposition 6. In the case of motions of an unbalanced tripod which are close to translational, the friction moment is of the first order of smallness with respect to the angular velocity. In the case when

$$(\mathbf{Jk}, \mathbf{k} - \mu \mathbf{i}) < 0 \tag{5.11}$$

friction leads to an increase in the absolute magnitude of the angular velocity, which is indicative of the instability of translational motions with the given parameters. In the case when inequality (5.11) has the opposite sign, friction leads to a retardation of the rotation.

6. Rotational motions of a tripod

In the other limiting case when $\upsilon = 0$, we obtain the following expressions for the velocities of the contact points:

$$\boldsymbol{\upsilon}_s = \omega \mathbf{k} \times \mathbf{r}_s^0, \quad s = 1, 2, 3 \tag{6.1}$$

In the case when $\omega > 0$, formulae (4.1) take the form

$$\omega^2(\mathbf{e}_n \times \mathbf{k}, \mathbf{J}\mathbf{k}) = \sum N_s\left(\mathbf{e}_n, \mathbf{r}_s^0 \times \mathbf{k} - \mu h\frac{\mathbf{r}_s^0}{|\mathbf{r}_s^0|} - \mu|\mathbf{r}_s^0|\mathbf{k}\right) \tag{6.2}$$

(the case when $\omega < 0$ can be described by the same formulae if the value of μ is replaced by the opposite value).

We will first discuss the case of a balanced tripod. Putting $\mathbf{e}_1 = \mathbf{i}$, $\mathbf{e}_2 = \mathbf{j}$, we transform system (6.2) to the form

$$\sum N_s\left(y_s - \mu h\frac{x_s}{\rho_s}\right) = 0, \quad \sum N_s\left(x_s + \mu h\frac{y_s}{\rho_s}\right) = 0; \quad \rho_s = |\mathbf{r}_s^0| \tag{6.3}$$

When account is taken of formulae (6.1), these equations are equivalent to the following formulae

$$\sum x_s N_s = -hT_X, \quad \sum y_s N_s = -hT_Y, \tag{6.4}$$

where the sums on the left-hand sides are equal to the coordinates of the centre of pressure.

We will now investigate under which conditions the equality $T = 0$ is possible. By virtue of formulae (6.4), this equality is equivalent to the system

$$\sum N_s x_s = \sum N_s y_s = \sum N_s\frac{x_s}{\rho_s} = \sum_{s=1}^{3} N_s\frac{y_s}{\rho_s} = 0 \tag{6.5}$$

The linear system (6.5), which is homogeneous in N_s, has a solution $N_s > 0$ only in the case when $\rho_1 = \rho_2 = \rho_3$, that is, the point G' is the centre of the circle circumscribed around the triangle $C_1C_2C_3$. Moreover, there are two limiting cases: the point G' coincides with the middle of one of the sides of the triangle or with its vertex (in this case one or two of the quantities N_s are equal to zero). In all these cases, the normal load in the motion is distributed in the same way as in the case of a smooth support and the existence criterion (Proposition 1) is satisfied. The following assertion is proved.

Proposition 7. In the case of a balanced tripod, rotational motions are possible in three cases: when the point G' coincides with the centre of a circle circumscribed around the supporting triangle, with the middle of one of its sides or with a vertex. Here, the principal vector of the friction forces is equal to zero and their principal moment is independent of the angular velocity.

Remark 1. If the conditions of Proposition 7 are not satisfied, then "pure" rotation of the tripod is impossible, since the friction forces lead to motion of the centres of mass. By virtue of equality (6.2), the principal vector of these forces and, also, their principal moment are independent of the magnitude of the angular velocity but depend on its sign.

Remark 2. In the case when $h = 0$ (a plate), the right-hand sides in formulae (6.4) are equal to zero and the normal load distribution is independent of the form of motion of the body. When the direction of rotation changes, the quantities $\mathbf{T}$ and $\mathbf{M}_T$ change into the opposite quantities and, in the general case, $\mathbf{T} \neq 0$.

Example. Suppose

$$\mathbf{r}_1^0 = (1, 0), \quad \mathbf{r}_2^0 = (0, 2), \quad \mathbf{r}_3^0 = (-1, -2)$$

and the weight of the tripod is equal to unity. Under these conditions, the point G' lies at the intersection of the medians of the triangle $C_1C_2C_3$ and therefore state at rest

$$N_1 = N_2 = N_3 = 1/3$$

We initially assume $h = 0$. Then the static normal load distribution is also maintained in the dynamics. Using formulae (4.9) and (6.1), for the case when $\omega > 0$ we find

$$T_X = \mu\sum N_s\frac{y_s}{\rho_s} = \frac{1}{3}\mu\left(1 - \frac{2}{\sqrt{5}}\right), \quad T_Y = -\mu\sum N_s\frac{x_s}{\rho_s} = -\frac{1}{3}\mu\left(1 - \frac{1}{\sqrt{5}}\right)$$
$$M_T = -\mu\sum N_s\rho_s = -\frac{1}{3}\mu(3 + \sqrt{5}) \tag{6.6}$$

Suppose $h > 0$ and the tripod is balanced. System (6.3) takes the form

$$N_1 + N_2 + N_3 = 1, \quad -\gamma N_1 + 2N_2 + \left(-2 + \frac{\gamma}{\sqrt{5}}\right)N_3 = 0$$
$$N_1 + \gamma N_2 - \left(1 + \frac{2}{\sqrt{5}}\gamma\right)N_3 = 0, \quad \gamma = \mu h\frac{\omega}{|\omega|} \tag{6.7}$$

Solving system (6.7), it is then possible to determine T_X, T_Y and M_T using formulae (6.6). In particular,

$$T_X = 0.09\mu, \quad T_Y = -0.10\mu, \quad M_T = -1.82\mu \ \text{ when } \gamma = 1$$

$$T_X = 0.04\mu, \quad T_Y = 0.16\mu, \quad M_T = 1.75\mu \text{ when } \gamma = -1$$

(the second case corresponds to the opposite direction of rotation).

The case of an unbalanced tripod differs from that the one considered in that, in Eqs (6.3), the right-hand sides are proportional to ω^2. Consequently, the normal load distribution as well as the friction forces will not only depend on the direction of the angular velocity but also on its modulus.

We will investigate the conditions under which a tripod can execute "pure rotations". We determine the vectors $\mathbf{e}_n$ in Eqs. (6.2) using formulae (4.7) and put $\mathbf{T}=0$. By analogy with system (6.5), we obtain the system

$$\sum N_s y_s = \sum N_s \frac{x_s}{\rho_s} = \sum N_s \frac{y_s}{\rho_s} = 0$$
$$\sum N_s(\alpha x_s + \beta\mu\rho_s) = 0 \tag{6.8}$$

The first three equations of (6.8) constitute a linear algebraic system in $\mathbf{N}_i$ and the existence of non-zero solutions is equivalent to degeneracy of the matrix of the system, that is,

$$\det\|\bar{x}_s \ \bar{y}_s \ \bar{y}_s\rho_s\| = 0 \tag{6.9}$$

where $\bar{\mathbf{x}}_s = (x_1, x_2, x_3)^T$ etc. Satisfaction of equality (6.9) can be achieved by rotating the supporting triangle about the GZ axis by a suitable angle φ, which is equivalent to the opposite rotation of the axes of the coordinates $G'XY$. In the new system $G'X^*Y^*$, the horizontal component of the $\mathbf{Jk}$ vector lies on the ordinate axis. The relation between the two systems of coordinates is described by the formulae

$$x^* = x\cos\varphi - y\sin\varphi, \quad y^* = x\sin\varphi + y\cos\varphi \tag{6.10}$$

Instead of x_s and y_s, we substitute the new coordinates (6.10) into equality (6.9), keeping the quantities ρ_s unchanged. After simple transformations of the determinant, we obtain the equation in the angle φ

$$\det\|\bar{x}_s \ \bar{y}_s \ \bar{y}_s\rho_s\|\cos\varphi + \det\|\bar{x}_s \ \bar{y}_s \ \bar{x}_s\rho_s\|\sin\varphi = 0 \tag{6.11}$$

If just one of the determinants in formula (6.11) is non-zero, this equation has two roots in the interval $\varphi \in [0, 2\pi)$. By successively substituting these roots into formulae (6.10), we find the direction of the horizontal component of the vector $\mathbf{Jk}$ and the distribution N_s. The last equation of (6.8) then enables us to find the parameters α and β, defining the angle between $\mathbf{Jk}$ and $\mathbf{k}$, uniquely. At the same time, the inequality

$$\sum N_s x_s < 0 \tag{6.12}$$

indicates the possibility of rotation with a positive angular velocity ω. If inequality (6.12) has the opposite sign, the angular velocity of rotation is negative. If the left-hand side in formula (6.7) is equal to zero, then $\beta=0$, that is, the tripod is balanced.

The equality to zero of both the determinants in formula (6.11) means that the configuration of the supporting points is described by one of the alternatives in Proposition 7. In this case, we also obtain $\beta=0$ in the last formula of (6.7).

Moreover, it is necessary to show that the normal load distribution is permissible. Since

$$\mathbf{T} = \sum N_s \mathbf{t}_s = 0$$

Proposition 2 can be used for this: for permissible solutions the triangle $C_1C_2C_3$ must contain the point G' shifted by an amount which is opposite to the horizontal component of the vector $\omega^2 P^{-1}\mathbf{Jk}$.

We will now assemble the results which have been obtained.

Proposition 8. In the case of an unbalanced tripod with an arbitrary geometry (determined by the vectors r_s), rotational motions only exist for two directions of the vector $\mathbf{Jk}$, which determines the distribution of the masses. These directions lie in one vertical plane and the rotations of the tripod in opposite directions correspond to them. At the same time, the rotation velocity lies in a finite interval which is determined from the condition that the normal reactions are non-negative.

Remark. If the conditions of Proposition 8 are not satisfied, then "pure" rotation of the tripod is impossible. By virtue of equality (6.2), the principal vector of the friction forces, as well as the principal moment, depend both on the magnitude of the angular velocity as well as on its sign.

Example. In the case of the tripod from the previous example, it follows from Proposition 8 that rotational motions are only possible in the case of an imbalance. In Eq. (6.11), we have

$$\det\|\bar{x}_s \ \bar{y}_s \ \bar{y}_s\rho_s\| = 8 - 4\sqrt{5}, \quad \det\|\bar{x}_s \ \bar{y}_s \ \bar{y}_s\rho_s\| = 2 - 2\sqrt{5}$$

whence we obtain

$$\operatorname{tg}\varphi = (\sqrt{5} - 3)/2 \tag{6.13}$$

The roots of Eq. (6.13) $\varphi_1 = -21^\circ$, $\varphi_2 = 159^\circ$ determine the position of the $G'X^*$ axis, which is perpendicular to the vector $\mathbf{Jk}$ (Fig. 2). When account is taken of the equality $N_1 + N_2 + N_3 = 1$, the solution of system (6.8) leads, for both of the values φ_1 and φ_2, to the distribution

$$N_1 = 0.19, \quad N_2 = 0.38, \quad N_3 = 0.43 \tag{6.14}$$

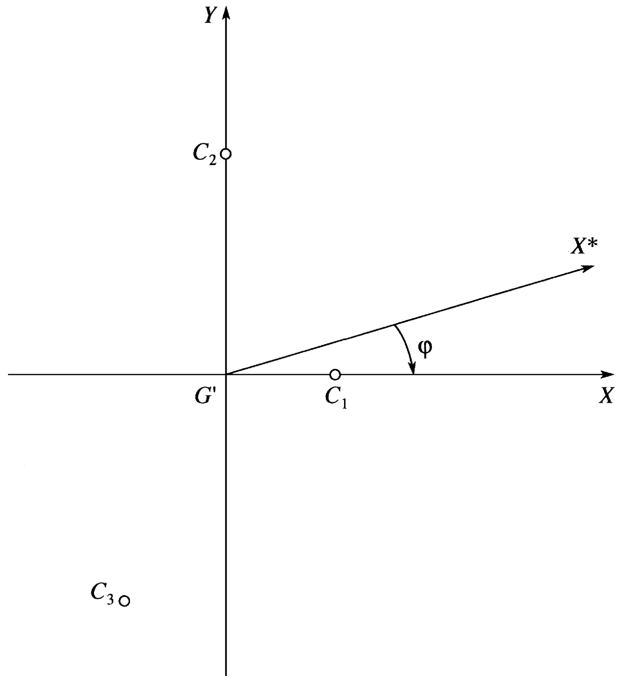

Fig. 2.

Note that the magnitude of M_T, corresponding to distribution (6.14), exceeds the friction moment for a balanced plate by 9%.

Substituting the values (6.14) into the last equation of (6.8), we determine the angle θ between **Jk** and the vertical:

$$\theta = -\arccos\frac{\mu\sum\rho_s N_s}{\sum x_s^* N_s} \approx \operatorname{arctg} 5\mu \tag{6.15}$$

It can be seen that the angle θ decreases as the coefficient of friction increaes and it is close to a right angle for small values of μ, which characterizes a high degree of imbalance of the body.

Acknowledgement

This research was financed by the Russian Foundation for Basic Research (08-01-00718).

References

1. Contensou P. Couplage entre frottement de glissement et frottement de pivotement dans la teorie de la toupie // Kreiselsprobleme. Gyrodynamics. Symp. Celerine, 1962. Derlin etc.: Springer, 1963.
2. Ishlinskii AYu, Sokolov BN, Chernous'ko FL. The motion of plane bodies with dry friction. *Izv Akad Nauk SSSR, MTT* 1981;**4**:17–28.
3. Zhuravlev VF. The model of dry friction in the problem of the rolling of rigid bodies. *Prikl Mat Mekh* 1998;**62**(5):762–7.
4. Zhuravlev VF. Frictional relationships in the case of a combination of sliding and curling. *Izv Ross Aklad Nauk MTT* 2003;**4**:81–8.
5. Kireenkov AA. The Motion of a homogeneous rotating disc along a plane under conditions of combined friction. *Izv Ross Akad Nauk MTT* 2002;**1**:60–7.
6. Johnston GW. The dynamics of curling stone. *Can Aeronaut Space J* 1981;**27**(2):144–60.
7. Samsonov VA. Friction accompanying the sliding and curling of a body. *Vestnik MGU Ser 1 Mat Mekh* 1981;**2**:76–8.
8. Denney M. Curling rock dynamics. *Canadian J Phys* 1998;**76**(4):295–304.
9. Penner AR. The physics of sliding cylinders and curling rocks. *Amer J Phys* 2001;**69**(3):332–9.
10. Rozenblat GM. *Dynamical Systems with Dry Friction.* Moscow; Izhevsk:RKhD; 2006.
11. MacMillan WD. *Dynamics of Rigid Bodies*. New York: McGraw-Hill; 1936.
12. Appell P. *Traite de Mecanique Rationell.* Paris: Gauthier-Villars; 1953.

Translated by E.L.S.